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1995 J. Phys. A: Math. Gen. 28 7287

(http://iopscience.iop.org/0305-4470/28/24/024)

J. Phys. A: Math. Gen. 28 (1995) 7287–7297. Printed in the UK

Elliptic eigenstates for the quantum harmonic oscillator

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Received 28 July 1995

Abstract. A new family of stationary coherent states for the two-dimensional harmonic oscillator is presented. These states are coherent in the sense that they minimize an uncertainty relation for observables related to the orientation and the eccentricity of an ellipse. The wavefunction of these states is particularly simple and well localized on the corresponding classical elliptical trajectory. As the number of quanta increases, the localization on the classical invariant structure is more pronounced. These coherent states give a useful tool to compare classical and quantum mechanics and form a convenient basis to study weak perturbations.

1. Introduction

It is well known that the notion of a coherent state for a quantum system was first proposed by Schrödinger in 1926 (see [1]), immediately after the birth of the quantum theory. The first modern application and development were made by Glauber in 1963 (see references in [2]), who introduced the term coherent state (CS), and initiated this important field of study now used in many areas of physics.

In quantum mechanics, the CS makes a natural connection with the dynamical group of the system and provides a phase space on which a classical mechanics can be defined. These properties have been extensively used for the phase-space representation of quantum states by the generalized Husimi distributions and for the study of the semiclassical approximation, especially in the case where the classical mechanics is mixed, regular and chaotic [3, 4]. Indeed the stationary eigenstates of a quantum system are expected to reflect the classical invariant structures.

In this paper, we illustrate this latter fact for the simple and well known case of a two-dimensional isotropic harmonic oscillator where the classical Hamiltonian can be simply expressed in terms of the generators of its dynamical group, $SU(2)$. The geometric discussion of the classical meaning of these quantities leads, in the quantum counterpart, to the construction of the CS of $SU(2)$, which are stationary and localized on the classical trajectories in the two-dimensional configuration space. This behaviour is of course different from the standard CS which are time-dependent and 'follow' the classical particle's motion.

For example, in the Coulombic case, the description of an electron in a Rydberg atom is based on the construction of a wavepacket representing the particle aspect [5, 6] (time-dependent CS). Such a state cannot be construct directly and one must construct the wavepacket by a superposition of time-independent eigenstates of various energies which represent the classical elliptic trajectories. Nevertheless, this construction does not lead to a well localized wavepacket [7, 8].

0305-4470/95/247287+11$19.50

Here we adopt a heuristic presentation: if the notion of trajectory is not a valid concept in quantum mechanics, due to the Heisenberg uncertainty relationships, we think that it would be illuminating to see that the stationary states are, in fact, related to some classical invariant, and that the CS offer a basic tool for expressing this analogy [9].

This paper is organized as follows. Section 2 gives the more compact form of these coherent states using complex coordinates. These coordinates are connected to the Cartesian (real) ones by a parameter associated to the eccentricity of the ellipse. We also give some visualizations of the probability density which clearly exhibit an elliptical shape and Bohr's quantization principle. Then, we recall the symmetry group of the classical two-dimensional harmonic oscillator in section 3. In section 4, we construct from the classical properties the CS of $SU(2)$. In section 5, we identify the quantum states built in section 2 with the CS of $SU(2)$. Section 6 is devoted to the relation between these *elliptical* CS and the usual two-dimensional CS. A discussion on the interest of these elliptical CS for weak perturbation is presented in the conclusion.

2. Complex representation of elliptic states

We consider a particle in a two-dimensional isotropic harmonic oscillator. Let η be a parameter such that $0 \leqslant \eta < \pi/4$ and consider the following complex canonical transformation:

$$\begin{aligned} X &= \frac{1}{\sqrt{\cos 2\eta}}(x\cos\eta + \mathrm{i}y\sin\eta) \\ Y &= \frac{1}{\sqrt{\cos 2\eta}}(-\mathrm{i}x\sin\eta + y\cos\eta) \end{aligned} \tag{1}$$

where x and y are the usual dimensionless Cartesian coordinates. The Hamiltonian in both bases has the same quadratic form:

$$H = \tfrac{1}{2}(x^2 + y^2 + {p_x}^2 + {p_y}^2) = \tfrac{1}{2}(X^2 + Y^2 + {P_X}^2 + {P_Y}^2). \tag{2}$$

Therefore any wavefunction $\Phi_{n_X}(X)\Phi_{n_Y}(Y)$ is a solution of the stationary Schrödinger equation where $\Phi_n(z) = \left(n!/\sqrt{\pi}2^n\right)^{1/2}\exp(-z^2/2)H_n(z)$ denotes the usual eigenfunctions of the one-dimensional oscillator associated with the eigenvalues $n + \frac{1}{2}$ [10] (H_n is a Hermite polynomial of order n). Of particular interest are functions defined as

$$\Psi_{n\eta}(x, y) = (\cos 2\eta)^{n/2}\Phi_n(X)\Phi_0(Y). \tag{3}$$

They represent normalized stationary states of (2) with the energy $n + 1$. Because Hermite polynomials of order n have n real zeros, $\Psi_{n\eta}(x, y)$ has only n nodes located on the Ox axis. Consequently these functions have less zeros than the usual ones, which are the tensorial product of n_x quanta in the Ox-direction and n_y quanta in the Oy-direction, i.e. $n_x + n_y = n$ lines of zeros on the plane. The probability density $|\Psi_{n\eta}|^2$ is peaked on the classical orbit:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where a and b are defined by $n + 1 = a^2/2 + b^2/2$ and we introduce the eccentricity $e = \sqrt{1 - b^2/a^2} = \sqrt{1 - \tan^2\eta}$. The three sub-figures (*a*)–(*c*) of figure 1 show the probability densities $|\Psi_{10\eta}|^2$ for three elliptic states with $n = 10$ and $\eta = 45°$, 30° and 20°, illustrating geometrically Bohr's correspondence principle and the classical areal velocity law: the classical particle spends more time near the two apogees of the motion, and therefore the quantum density has a peak at these points (see the discussion in [9]).

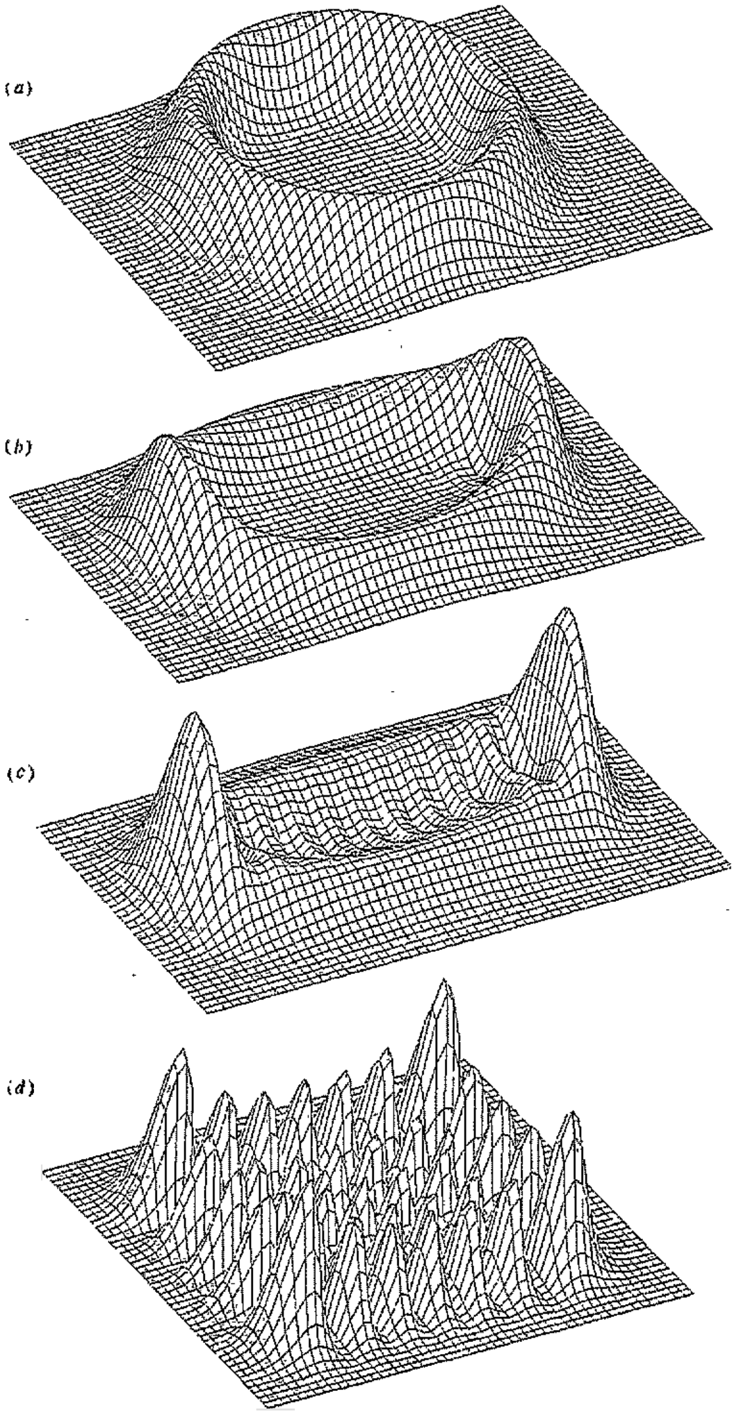

Figure 1. The probability densities $|\Psi_{10\eta}|^2$ associated with stationary coherent elliptic states with $n = 10$ quanta and: (*a*) $\eta = 45°$ the circular state with maximum orbital momentum, (*b*) $\eta = 30°$, eccentricity $e = 0.81$ and (*c*) $\eta = 20°$, eccentricity $e = 0.93$. It is worthwhile to note the ten nodes on the major axis are degenerated at the origin for the circular state. (*d*) A standard eigenstate with six quanta in Ox and 4 in Oy.

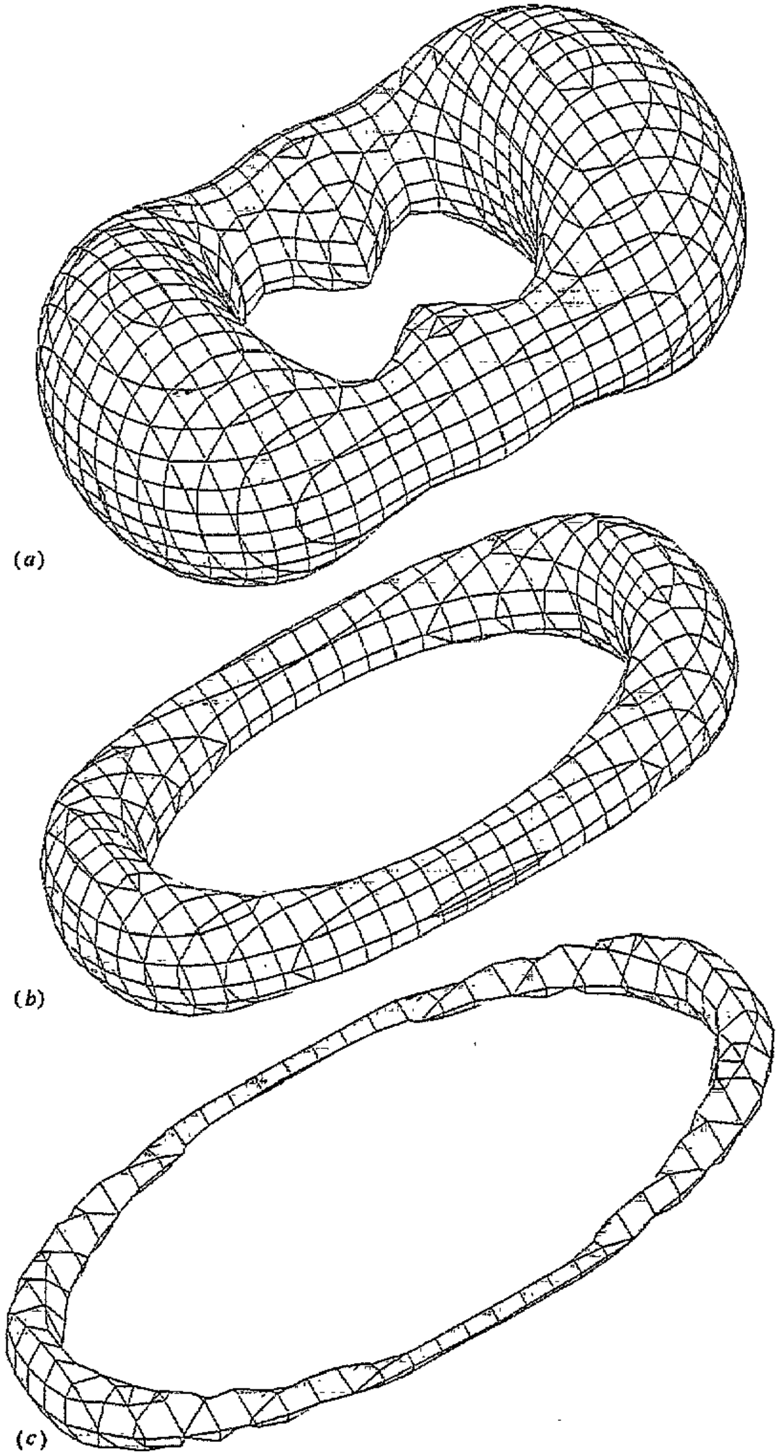

Figure 2. The 3D elliptic states ($e = 0.93$) with zero quanta on Oz. The iso-surface of $\left|\Psi_{n\eta}\right|^2$ defines a volume corresponding to a probability $\frac{1}{2}$ of finding the particle within. These figures illustrate the Bohr correspondence principle. (a) $n = 2$, (b) $n = 10$, (c) $n = 50$.

Figure 1(a) is a representation of the eigenstate which corresponds to the maximal z-component of the angular momentum. It is of course circular. Figure 1(d) shows the probability density for a standard state of the same number of quanta (six quanta in Ox and four in Oy, i.e. ten lines of zeros). Figure 2 presents a perspective view of three iso-density surfaces for the three-dimensional harmonic oscillator with 0 quanta in the Oz-direction and 2, 10 and 50 quanta in the (Ox, Oy) plane. These surfaces are calculated so that the particle has a 50% probability to be inside the volume enclosed by the surface. We remark that a large quantum number is not necessary for the localization of the particle density on the classical trajectory (Bohr's principle).

Following the standard procedure, we introduce from the canonical transformation (1), the creation and annihilation operators:

$$A_X^{\pm} = \frac{1}{\sqrt{2}}(X \pm \mathrm{i}P_X) = \frac{1}{\sqrt{\cos 2\eta}}\left(a_x^{\pm}\cos\eta + \mathrm{i}a_y^{\pm}\sin\eta\right).$$

They satisfy the usual commutation relations. The state associated with wavefunction (3) can be written as

$$|\Psi_{n\eta}\rangle = \sqrt{\frac{(\cos 2\eta)^n}{n!}}(A_X^{+})^n\,|0\rangle \tag{4}$$

where $|0\rangle$ is the ground state of the oscillator. The factor $\cos 2\eta$ comes from the fact that A_X^+ is not the adjoint of A_X^-.

3. The group $SU(2)$ of the classical harmonic oscillator

Before introducing the CS, let us recall some results about the classical 2D-isotropic harmonic oscillator. In order to provide the *ad hoc* construction and define notations, let us consider the Hamiltonian (2) in the Cartesian coordinates. The group structure of this system is $SU(2)$ [11], which is defined by the permutations between the generators

$$\{S_i, S_j\} = \epsilon_{ijk}S_k \tag{5}$$

where ϵ_{ijk} is the antisymmetrical tensor, $\{.,.\}$ denotes the Poisson bracket and S_i are three constants of motion:

$$S_1 = \tfrac{1}{4}(x^2 + p_x^2 - y^2 - p_y^2) \qquad S_2 = \tfrac{1}{2}(p_x p_y + xy) \qquad S_3 = \tfrac{1}{2}(xp_y - yp_x)\,. \tag{6}$$

Due to the number of degrees of freedom, H and S_i are not independent, and direct calculation gives

$$H^2 = 4S^2 \tag{7}$$

where $S^2 = {S_1}^2 + {S_2}^2 + {S_3}^2$. Thus, the set S_i characterizes the geometrical configuration of the trajectory. Let us describe the meaning of these quantities. For a given trajectory they correspond to a unique point on a sphere, called *the Bloch sphere* (figure 3) of radius $S = H/2 = E/2$ (E is the energy). We define the spherical coordinates relative to S_3:

$$S_1 = S\sin\theta\cos\varphi \qquad S_2 = S\sin\theta\sin\varphi \qquad S_3 = S\cos\theta\,.$$

Let us now consider the ellipse $\tilde{x}^2/a^2 + \tilde{y}^2/b^2 = 1$, $a > b$, with

$$\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix} = \begin{pmatrix}\cos\frac{\varphi}{2} & \sin\frac{\varphi}{2}\\ -\sin\frac{\varphi}{2} & \cos\frac{\varphi}{2}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}.$$

The angle $\frac{\varphi}{2}$ represents the orientation of the major axis of the ellipse. We defined η such that $a = \sqrt{2E}\cos\eta$ and $b = \sqrt{2E}\sin\eta$, where η is related to the eccentricity by

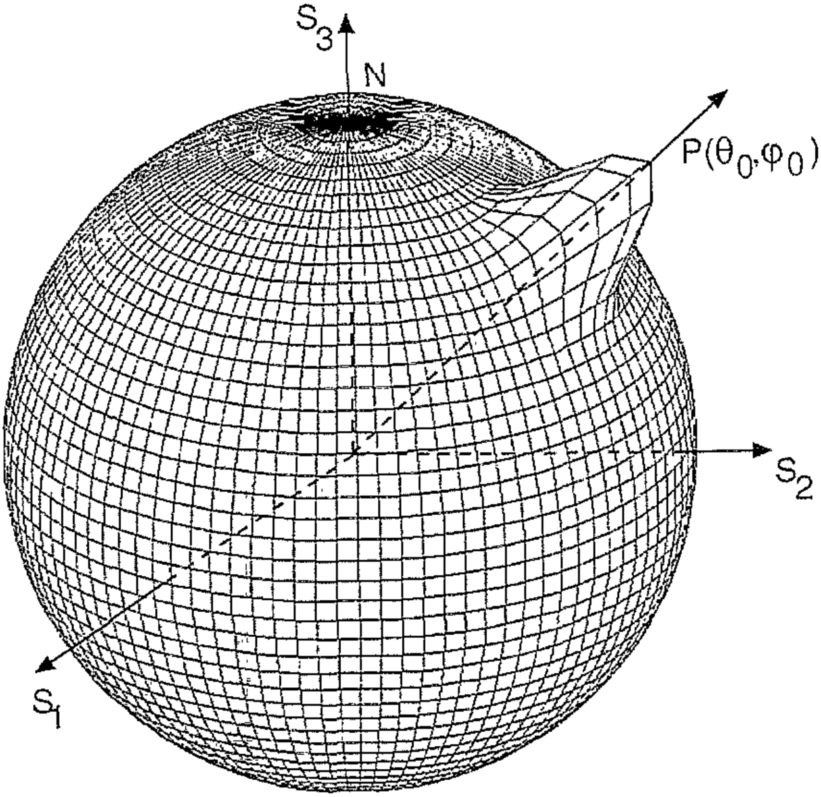

Figure 3. The Bloch sphere: an alternative representation of the phase space for the two-dimensional isotropic oscillator. The coordinates θ and φ give the characteristics (orientation, eccentricity, major axis) of a classical ellipse associated with the corresponding coherent state. The surface has been constructed by deforming the Bloch sphere radially by a quantity proportional to the squared scalar product of a particular coherent elliptic state $|n = 200; \theta_0, \varphi_0\rangle$ with the CS corresponding to the direction (θ, φ).

$e = \sqrt{1 - \tan^2 \eta}$. Direct calculations give $\theta = \frac{\pi}{2} - 2\eta$. The interpretation of S_1 and S_2 becomes more evident if we note that $S_1 = \frac{a^2-b^2}{4} \cos\varphi$ and $S_2 = \frac{a^2-b^2}{4} \sin\varphi$ so that the focal distance $f = 4({S_1}^2 + {S_2}^2)^{1/4}$. The generator S_3 is clearly half of the z-component of the classical momentum L_z, the rotation generator in the plane.

The set S_i can also be considered as a set of generating functions of canonical transformations [11] which correspond to rotations on the Bloch sphere. Therefore any ellipse can be deduced from another one by a rotation on the sphere.

For example, using the previous expressions of S_1 and S_2, a rotation of φ about S_3 will keep a, b, f and E constant and change only the orientation of the ellipse. On the other hand, rotations about S_1 or S_2 will, in general, modify both the eccentricity and the orientation, except when $\varphi = 0$, where rotations about S_2 will change only the eccentricity alone, as is also the case about S_1 for $\varphi = \frac{\pi}{2}$.

If we now consider the motion of a particle on the classical trajectory, we need only one more quantity to describe its position, namely the time expressed in a suitable dimension. This gives a geometrical parametrization of the phase space for this system by the set S_i and t.

4. Coherent states of $SU(2)$

The quantum counterpart of this system is expressed by the same relations (2), (5)–(7) using commutators instead of Poisson brackets and $i\epsilon_{ijk}$ instead of ϵ_{ijk}. We set $\hbar = 1$ and denote the classical generators and the quantum operators by the same letters. Due to the commutators the relation (7) becomes

$$H^2 = 4S^2 + 1 . \tag{8}$$

The eigenvalues of S^2 are $s(s+1)$ where the principal quantum number $n = 2s$. The $2s+1$ eigenvalues of S_i are $-s, -s+1, \ldots, s$. Because of the classical origin of S_1, S_2, S_3, the quantum states 'localized' on the elliptic orbits that we seek will, in some sense, minimize the dispersion of the measurement of these observables. But the commutation relations given by the quantization of (5) leads to a kind of Heisenberg uncertainty relation which can be visualized on the Bloch sphere: the classical ellipse is associated with a single point on the sphere but the quantum state can at best minimize a solid angle around this point. Moreover, the non-zero lower bound has to be intrinsically defined, that is to say, invariant under the rotations (just as the Heisenberg relation $\Delta x \cdot \Delta p \geqslant 1$ is covariant under the Galileian transformations).

At first sight, the commutation relations lead to the condition $\Delta S_1 \Delta S_2 \geqslant |\langle S_3 \rangle|/2$ (or circular permutations), but this is not covariant because $|\langle S_3 \rangle|$ cannot define a uniform unit of area on the sphere. For example, this area will be null on the equator. The better choice is in fact the following [12]: given a small area around some point P on the Bloch sphere (see figure 3), let $\tilde{R}$ be one of the rotations which maps P onto N, the north pole defined by S_3. Then the rotated operators $\tilde{S}_3 = \tilde{R}^{-1} S_3 \tilde{R}$ satisfy the same commutation rules as the previous set and the uncertainty relations will be

$$\Delta(\tilde{S}_1)\Delta(\tilde{S}_2) \geqslant |\langle \tilde{S}_3 \rangle|/2 . \tag{9}$$

The quantity $\Delta(\tilde{S}_1)\Delta(\tilde{S}_2)$ measures the area centred at P. Note that (9) is now obviously covariant under the symmetry group of the sphere. It is well known that the CS of the Lie group $SU(2)$ will minimize the inequality (9). Results concerning their construction and their properties are essentially taken from [2, 13].

Let us briefly recall the construction for a general Lie group G: given a Hilbert space which gives an irreducible representation of G, any state $|\Psi_0\rangle$ in this space determines a subgroup H of G, the isotropic subgroup consisting of elements which leave $|\Psi_0\rangle$ invariant in the quantum sense, that is to say, up to a phase factor. The choice of this initial state is quite arbitrary, but it is interesting to take a certain *extremal state* (related to the symmetry properties of a physical system for example). Now the family of generalized CS is the set $\{\Omega\,|\Psi_0\rangle\,,\, \Omega \in G/H\}$ which has the same topology as the coset space G/H (dependent on $|\Psi_0\rangle$).

In our case, from (8), it is clear that the eigenspaces of H give an irreducible representation of $SU(2)$ where we can diagonalize both H and one of the S_i. We choose for the extremal state a totally circular and normalized one (see figure 1(a)): $|\Psi_0\rangle = |s, -s\rangle$ defined by

$$\begin{aligned} H\,|s,-s\rangle &= (2s+1)\,|s,-s\rangle \\ S^2\,|s,-s\rangle &= s(s+1)\,|s,-s\rangle \\ S_3\,|s,-s\rangle &= -s\,|s,-s\rangle . \end{aligned}$$

The coset space is isomorphic to the sphere, which is of course the Bloch sphere. The displacement operators Ω are rotations on the sphere in accord with the classical picture so that they will depend on two angles.

Thus each CS, $|n;\theta,\varphi\rangle = \Omega(\theta,\varphi)\,|s,-s\rangle$, will be associated with a point on the sphere and will be located within a probability distribution on the corresponding classical trajectory. In particular, $|s,-s\rangle$ corresponds to $|n=2s;\theta=\pi,\varphi\rangle$.

The dispersions of the measures of the rotated operators $\tilde{S}_1$ and $\tilde{S}_2$ are equal $\Delta(\tilde{S}_1) = \Delta(\tilde{S}_2) = \sqrt{s/2}$ and we are in the optimal situation where $\Delta(\tilde{S}_1)\Delta(\tilde{S}_2) = |\langle\tilde{S}_3\rangle|/2 = s/2$. The localization on the Bloch sphere can also be seen by the invariant relation $\Delta(S_1)^2 + \Delta(S_2)^2 + \Delta(S_3)^2 = s = n/2$, that is to say the area surrounding P changes as n whereas the total area of the Bloch sphere behaves like n^2.

We can expand the normalized CS using the eigenstates $|s,m\rangle$ of S^2 and S_3 with eigenvalues $s(s+1)$ and m:

$$\begin{aligned}|n=2s;\theta,\varphi\rangle &= \Omega(\theta,\varphi)\,|s,-s\rangle \\ &= \sum_{m=-s}^{s}\sqrt{\frac{(2s)!}{(s+m)!(s-m)!}}\left(\cos\frac{\theta}{2}\right)^{s+m}\left(\sin\frac{\theta}{2}\right)^{s-m}\mathrm{e}^{-\mathrm{i}(s+m)\varphi}\,|s,m\rangle\,. \end{aligned} \qquad (10)$$

They form an over-complete basis in each eigenspace of H:

$$Id = \frac{n+1}{4\pi}\int_{\substack{\text{Bloch}\\ \text{sphere}}} |n;\theta,\varphi\rangle\sin\theta\,\mathrm{d}\theta\mathrm{d}\varphi\,\langle n;\theta,\varphi|\,. \qquad (11)$$

In this integral, θ goes from 0 to π, i.e. from an anticlockwise circular orbit to a clockwise circular one.

The scalar product between two different CS associated with two different points on the sphere is simply expressed in terms of the angle Θ between the corresponding radial unit vectors:

$$\left|\langle n;\theta',\varphi'|n;\theta,\varphi\rangle\right|^2 = \left(\cos\frac{\Theta}{2}\right)^{2n}. \qquad (12)$$

This scalar product allows us to visualize the localization of a CS on the Bloch sphere. In figure 3, we report from the Bloch sphere and in the (θ,φ) direction, the squared scalar product of the corresponding CS with a particular one $|n;\theta_0,\varphi_0\rangle$. From equation (13) the characteristic angle associated with a CS behaves like $1/\sqrt{n}$ and the surface on the unit sphere behaves like $1/n$. The sphere supports n states, n being the dimension of the energy sub-space.

5. Elliptic creation operator

We will now show the identity between the elliptic states previously introduced and the coherent states by introducing elliptic creation operators. We can use the basis $|\cdot\rangle_+|\cdot\rangle_-$ of clockwise and anticlockwise circular polarization normalized states [10] with the corresponding creation operators a_+^+ and a_-^+ in order to give a new expression for the coherent states:

$$|s,m\rangle = \frac{1}{\sqrt{(s+m)!(s-m)!}}a_+^{+(s+m)}a_-^{+(s-m)}\,|0\rangle\,. \qquad (13)$$

Using equation (13) we rewrite the CS as

$$|n=2s;\theta,\varphi\rangle = \frac{1}{\sqrt{n!}}\left(a_+^+\cos\frac{\theta}{2}\mathrm{e}^{-\mathrm{i}\frac{\varphi}{2}} + a_-^+\sin\frac{\theta}{2}\mathrm{e}^{+\mathrm{i}\frac{\varphi}{2}}\right)^n|0\rangle = \frac{1}{\sqrt{n!}}(C_{\theta,\varphi}^+)^n\,|0\rangle$$

where $C_{\theta,\varphi}^+$ can be interpreted as a creation operator of one quantum, with a fractional distribution in each circular polarization and a phase shift between them. If $\theta = 0$ or

$\theta = \pi$, $|n;\theta,0\rangle = |s,\pm s\rangle$ then from (10) we obtain a totally circular state in accordance with the classical analysis. For other values, we obtain a non-isotropic deformation of the circle, i.e. an ellipse with a shape controlled by θ, which is related to the eccentricity. The operator $C^+_{\theta,0}$ is thus a kind of 'elliptical' creation operator.

We can construct the same CS from an extremal state of S_1 instead of S_3. Remarking that S_1 is proportional to the energy difference between the two linear modes along Ox and Oy (see equation (6)), the extremal state is then $\left|\Psi'_0\right\rangle = (a^+_y)^n/\sqrt{n!}\,|0\rangle$ which is the state with $n = 2s$ quanta in the mode Oy; the spherical angles (θ', φ') are now related to the S_1 axis. For $\varphi = 0$, we have $\varphi' = \pi/2$ and $\theta' = \pi/2 - \theta = 2\eta$, and in the same way, the rotation gives

$$|n;\theta,\varphi\rangle = \frac{1}{\sqrt{n!}}\left(a^+_x \cos\eta + \mathrm{i}a^+_y \sin\eta\right)^{2s}|0\rangle \tag{14}$$

the corresponding wavefunction reads

$$\langle x, y|n;\theta,\varphi=0\rangle = \sqrt{\frac{n!}{\pi 2^n}}\,\mathrm{e}^{-\frac{x^2+y^2}{2}}\sum_{p=0}^{n}\frac{(\cos\eta)^{n-p}(\mathrm{i}\sin\eta)^p}{p!(n-p)!}\,H_{\bar{n}-p}(x)H_p(y)\,. \tag{15}$$

Using the summation theorem for the Hermite polynomials [14] the sum can be re-cast as

$$\sum_{p=0}^{n}\frac{(\cos\eta)^{n-p}(\mathrm{i}\sin\eta)^p}{p!(n-p)!}\,H_{n-p}(x)H_p(y) = \frac{(\cos 2\eta)^{n/2}}{n!}H_n\left(\frac{x\cos\eta + \mathrm{i}y\sin\eta}{\sqrt{\cos 2\eta}}\right)$$

which allows us to identify the states introduced in section 2 to the CS. Rewriting $C^+_{\theta,0}$ in terms of the linear creation operators a^+_x and a^+_y (see equation (14)), we can see this operator as an elliptical creation operator of n quanta with a fractional distribution in each of the longitudinal modes which are in quadrature. We recognize then expression (4).

6. Relation with the standard CS

The previous CS were stationary by construction. Indeed the Heisenberg inequality $\Delta E \Delta t \geqslant 1$ is minimized by $\Delta E = 0$ and $\Delta t = \infty$, so that these states are totally delocalized in time. But this inequality can also be minimized by a finite localization both in E and t: such states will not be stationary. For $\Delta E \neq 0$, there is a family of Bloch spheres associated with the discrete values of E and each can 'support' states which minimize (9). Indeed, let us consider the superposition of elliptic CS with the same value of (θ, φ) but different energy. This superposition defines a mean classical ellipse on which the probability density will be localized; however, because $\Delta E \neq 0$, the ellipse will evolve with the time. Such a superposition is similar to the standard coherent state constructed on the Heisenberg group [13,2]. Here we briefly explain the relation with the standard time-dependent CS [10]. They form a complete set so that we can express $|n;\theta,0\rangle$ as a superposition of these states. The standard CS localized at the point (x, y, p_x, p_y) of the phase space has the following expansion:

$$|\alpha,\beta\rangle = \mathrm{e}^{-\frac{\alpha\alpha^* + \beta\beta^*}{2}}\sum_{n_x,n_y}\frac{\alpha^{n_x}\beta^{n_y}}{\sqrt{n_x!n_y!}}\,|n_x\rangle|n_y\rangle$$

with $\alpha = \frac{1}{\sqrt{2}}(x + \mathrm{i}p_x) = \rho\cos\lambda\exp(\mathrm{i}\phi_\alpha)$ and $\beta = \frac{1}{\sqrt{2}}(y + \mathrm{i}p_y) = \rho\sin\lambda\exp(\mathrm{i}\phi_\beta)$.

And the resolution of unity gives

$$|n;\theta,0\rangle = \int\frac{\mathrm{d}\alpha\,\mathrm{d}\beta}{\pi^2}\,|\alpha\beta\rangle\,\langle\alpha\beta|n;\theta,0\rangle$$

and direct calculations give $\langle\alpha\beta|n;\theta,0\rangle = (n!)^{-1/2}(\alpha^*\cos\eta + \mathrm{i}\beta^*\sin\eta)^n \exp(-\frac{\alpha\alpha^*+\beta\beta^*}{2})$. Using the expressions above for α and β, we rewrite this scalar product as

$$\langle\alpha\beta|n;\theta,0\rangle = \mathrm{e}^{-\rho^2/2}\frac{\rho^n}{\sqrt{n!}}(\cos\lambda\cos\eta\,\mathrm{e}^{-\mathrm{i}\phi_\alpha} + \sin\lambda\sin\eta\,\mathrm{e}^{\mathrm{i}(\frac{\pi}{2}-\phi_\beta)})^n . \qquad (16)$$

In the classical limit ($E = n+1 \to \infty$), the foliation by the Bloch spheres is a quasi-continuum since their radii are proportional to n and two consecutive spheres are distant by unity.

In this limit, the behaviour of (16) will select some of the α and β in the superposition. Actually, the parentheses has a norm less or equal to 1 and it will be non-zero in the semiclassical regime only if it has exactly a unit norm: therefore we must have $\lambda = \eta$ and $\phi_\alpha = \phi_\beta - \frac{\pi}{2}$, this gives the eccentricity expected and the quadrature phase between the motion on the two modes. Moreover, in this limit, $\rho^n \exp(-\rho^2/2)$ is strongly peaked around $\sqrt{n}$ which is exactly what we want for the energy. Note that the well known fact that the wavepacket for the standard CS does not spread is explained by the isochronism of the oscillations (i.e. equidistant energy levels). This is not true for the Coulomb potential where spreading of wavepackets occurs with revivals [15, 7].

7. Discussion and conclusion

In classical mechanics, a state of the isotropic two-dimensional harmonic oscillator is usually described by a point moving on an iso-energy surface in a four-dimensional phase space. It is equivalent to choosing the usual position-impulsion coordinates or the (S_1, S_2, S_3) which define an ellipse and a fourth coordinate (angle or time) which gives the position of the particle on the ellipse. Instead of S_1, S_2 and S_3, one can also use the spherical coordinates θ and φ on the Bloch sphere whose radius is half the energy.

Within first-order perturbation theory, the energy remains constant and the point describing the state of the system in these coordinates will drift slowly on the sphere with a characteristic time that is lower than the period of the non-perturbative periodic motion.

In the quantum description, the stationary coherent states of energy $n+1$ that we constructed in spherical coordinates are more natural than the standard ones: they correspond to a maximum in the localization of the distribution of measurements of the S_1, S_2 and S_3 operators on the sphere. They delimit on it a circular area of size n so that we need only n to cover the whole sphere. These states depend continuously on two parameters which define a point on the sphere. They provide a complete quasi-orthogonal basis. The corresponding wavefunctions are very simple because they use only one Hermite polynomial. Unlike the standard stationary states, their density probability has very little structure and is localized on the classical ellipse. They provide a natural link between the classical and quantum aspects of the harmonic oscillator.

A weak perturbation will lift the degeneracy and we must then diagonalize the perturbation in the $(n+1)$-energy eigenspace. The elliptic states are particularly convenient for this. One can represent on the sphere the squared scalar product of the elliptic state with the coherent state corresponding to each point of the sphere. For a regular classical trajectory the product will be significant only for a small area surrounding the trajectory and the expansion of the stationary state requires only a few CS. These states will be taken as centred and regularly spaced on the closed curve of this *trajectory*. The characteristic distance between the CS being $\sqrt{n}$, we need only $\sqrt{n}$ of the n complete basis to describe the perturbed state. We can also obtain from the periodic classical *trajectory* the semiclassical energy by the WKB quantization on the sphere. For classically chaotic motions, from the

KAM theorem, the chaotic region will be limited on the sphere and again, only a fraction of the complete basis of CS will be necessary. The CS will be well suited to visualizing the time dependence of the initial state. For small times, one can expand the true time-dependant state on a few CS which evolve according to the Hamiltonian flow.

Acknowledgment

We would like to thank F Faure for helpful discussions.

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